Fair Lawn

Public Schools

Fair Lawn, NJ

Adopted August

Multivariable Calculus

2015

Revised August 2015
Developed August 2012

The Multivariable course has been designed for students who have completed Advanced Calculus BC. This course aligns to a college level Multivariable Calculus Course and will prepare students for Differential Equations.

Multivariable Calculus

Fair Lawn School District

Committee Credits
Multivariable Calculus Team

Victoria Velasco, Teacher
Lauren Gimon, Supervisor

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Multivariable Calculus

I. Course Synopsis

Multivariable Calculus is the final course in the Fair Lawn High School accelerated mathematics course sequence. Students of Multivariable Calculus will learn to extend the tools they learned in AP Calculus BC to help understand more complex problems that cannot be addressed using the techniques of single-variable Calculus. In their first year of Calculus, they were limited to problems in two-dimensions, or three-dimensional shapes that could be easily seen as manipulations of two-dimensional shapes. Multivariable Calculus opens up the greater reality of three-dimensional space and of functions of more than one variable, giving the students tools to better understand a three-dimensional world. This course will prepare students for further study in all branches of higher mathematics, science and related fields.

Technology and laboratory work are used as appropriate to reinforce these approaches Topics included in this course are: Vectors and the Geometry of Space, Vector-Valued Functions, Functions of Several Variables, Multiple Integration, and Vector Analysis. Vectors have many applications in geometry, physics, engineering, and economics. The student builds on many of the ideas of calculus of a single variable to calculus of several variables.

II. Philosophy & Rationale

Multivariable Calculus will utilize a variety of instructional formats that will include lecture, discussion, group exploration and discovery, independent research, and visualization using the graphing calculator. Students will be encouraged to become active learners by challenging them to apply their prior knowledge and experience in new and increasingly more difficult situations. Individual work, small group, chalkboard demonstrations, and problem solving sessions will provide opportunities for students to exchange ideas and develop their ability to communicate and reason mathematically. Assessment of student achievement each marking period will be determined by a minimum of seven evaluations comprised of a combination of teacher-made examinations on major topics and quizzes, which will take forms to include oral, written, or through demonstration.

Teacher observations and evaluations of class work, homework, class participation, portfolios, and/or journals will be included. In addition, punctuality and a good attendance record are encouraged to avoid incomplete grades and/or unnecessary failures. The course will have a midyear and final examination that together will constitute a grade comparable in weight to a full marking period. The course will receive Advanced Placement weight.

COMMON CORE STATE STANDARDS

The Standards for Mathematical Practice describe varieties of expertise that mathematics educators at all levels should seek to develop in their students. These practices rest on important "processes and proficiencies" with longstanding importance in mathematics education. The first of these are the NCTM process standards of problem solving, reasoning and proof, communication, representation, and connections. The second are the strands of mathematical proficiency specified in the National Research Council's report Adding It Up: adaptive reasoning, strategic competence, conceptual understanding (comprehension of mathematical concepts, operations and relations), procedural fluency (skill in carrying out procedures flexibly, accurately, efficiently and appropriately), and productive disposition (habitual inclination to see mathematics as sensible, useful, and worthwhile, coupled with a belief in diligence and one's own efficacy).

CCSS.MATH.PRACTICE.MP1 - Make sense of problems and persevere in solving them.

Mathematically proficient students start by explaining to themselves the meaning of a problem and looking for entry points to its solution. They analyze givens, constraints, relationships, and goals. They make conjectures about the form and meaning of the solution and plan a solution pathway rather than simply jumping into a solution attempt. They consider analogous problems, and try special cases and simpler forms of the original problem in order to gain insight into its solution. They monitor and evaluate their progress and change course if necessary. Older students might, depending on the context of the problem, transform algebraic expressions or change the viewing window on their graphing calculator to get the information they need. Mathematically proficient students can explain correspondences between equations, verbal descriptions, tables, and graphs or draw diagrams of important features and relationships, graph data, and search for regularity or trends. Younger students might rely on using concrete objects or pictures to help conceptualize and solve a problem. Mathematically proficient students check their answers to problems using a different method, and they continually ask themselves, "Does this make sense?" They can understand the approaches of others to solving complex problems and identify correspondences between different approaches.

CCSS.MATH.PRACTICE.MP2 - Reason abstractly and quantitatively.

Mathematically proficient students make sense of quantities and their relationships in problem situations. They bring two complementary abilities to bear on problems involving quantitative relationships: the ability to *decontextualize*—to abstract a given situation and represent it symbolically and manipulate the representing symbols as if they have a life of their own, without necessarily attending to their referents—and the ability to *contextualize*, to pause as needed during the manipulation process in order to probe into the referents for the symbols involved. Quantitative reasoning entails habits of creating a coherent representation of the problem at hand; considering the units involved; attending to the meaning of quantities, not just how to compute them; and knowing and flexibly using different properties of operations and objects.

CCSS.MATH.PRACTICE.MP3 - Construct viable arguments and critique the reasoning of others.
Mathematically proficient students understand and use stated assumptions, definitions, and previously established results in constructing arguments. They make conjectures and build a logical progression of statements to explore the truth of their conjectures. They are able to analyze situations by breaking them into cases, and can recognize and use counterexamples. They justify their conclusions, communicate them to others, and respond to the arguments of others. They reason inductively about data, making plausible arguments that take into account the context from which the data arose. Mathematically proficient students are also able to compare the effectiveness of two plausible arguments, distinguish correct logic or reasoning from that which is flawed, and—if there is a flaw in an argument—explain what it is. Elementary students can construct arguments using concrete referents such as objects, drawings, diagrams, and actions. Such arguments can make sense and be correct, even though they are not generalized or made formal until later grades. Later, students learn to determine domains to which an argument applies. Students at all grades can listen or read the arguments of others, decide whether they make sense, and ask useful questions to clarify or improve the arguments.

CCSS.MATH.PRACTICE.MP4 - Model with mathematics.
Mathematically proficient students can apply the mathematics they know to solve problems arising in everyday life, society, and the workplace. In early grades, this might be as simple as writing an addition equation to describe a situation. In middle grades, a student might apply proportional reasoning to plan a school event or analyze a problem in the community. By high school, a student might use geometry to solve a design problem or use a function to describe how one quantity of interest depends on another. Mathematically proficient students who can apply what they know are comfortable making assumptions and approximations to simplify a complicated situation, realizing that these may need revision later. They are able to identify important quantities in a practical situation and map their relationships using such tools as diagrams, two-way tables, graphs, flowcharts and formulas. They can analyze those relationships mathematically to draw conclusions. They routinely interpret their mathematical results in the context of the situation and reflect on whether the results make sense, possibly improving the model if it has not served its purpose.

CCSS.MATH.PRACTICE.MP5 - Use appropriate tools strategically.
Mathematically proficient students consider the available tools when solving a mathematical problem. These tools might include pencil and paper, concrete models, a ruler, a protractor, a calculator, a spreadsheet, a computer algebra system, a statistical package, or dynamic geometry software. Proficient students are sufficiently familiar with tools appropriate for their grade or course to make sound decisions about when each of these tools might be helpful, recognizing both the insight to be gained and their limitations. For example, mathematically proficient high school students analyze graphs of functions and solutions generated using a graphing calculator. They detect possible errors by strategically using estimation and other mathematical knowledge. When making mathematical models, they know that technology can

enable them to visualize the results of varying assumptions, explore consequences, and compare predictions with data. Mathematically proficient students at various grade levels are able to identify relevant external mathematical resources, such as digital content located on a website, and use them to pose or solve problems. They are able to use technological tools to explore and deepen their understanding of concepts.

CCSS.MATH.PRACTICE.MP6 - Attend to precision.
Mathematically proficient students try to communicate precisely to others. They try to use clear definitions in discussion with others and in their own reasoning. They state the meaning of the symbols they choose, including using the equal sign consistently and appropriately. They are careful about specifying units of measure, and labeling axes to clarify the correspondence with quantities in a problem. They calculate accurately and efficiently, express numerical answers with a degree of precision appropriate for the problem context. In the elementary grades, students give carefully formulated explanations to each other. By the time they reach high school they have learned to examine claims and make explicit use of definitions.

CCSS.MATH.PRACTICE.MP7 - Look for and make use of structure.
Mathematically proficient students look closely to discern a pattern or structure. Young students, for example, might notice that three and seven more is the same amount as seven and three more, or they may sort a collection of shapes according to how many sides the shapes have. Later, students will see 7×8 equals the well-remembered $7 \times 5 + 7 \times 3$, in preparation for learning about the distributive property. In the expression $x^2 + 9x + 14$, older students can see the 14 as 2×7 and the 9 as $2 + 7$. They recognize the significance of an existing line in a geometric figure and can use the strategy of drawing an auxiliary line for solving problems. They also can step back for an overview and shift perspective. They can see complicated things, such as some algebraic expressions, as single objects or as being composed of several objects. For example, they can see $5 - 3(x - y)^2$ as 5 minus a positive number times a square and use that to realize that its value cannot be more than 5 for any real numbers x and y.

CCSS.MATH.PRACTICE.MP8 - Look for and express regularity in repeated reasoning.
Mathematically proficient students notice if calculations are repeated, and look both for general methods and for shortcuts. Upper elementary students might notice when dividing 25 by 11 that they are repeating the same calculations over and over again, and conclude they have a repeating decimal. By paying attention to the calculation of slope as they repeatedly check whether points are on the line through (1, 2) with slope 3, middle school students might abstract the equation $(y - 2)/(x - 1) = 3$. Noticing the regularity in the way terms cancel when expanding $(x - 1)(x + 1)$, $(x - 1)(x^2 + x + 1)$, and $(x - 1)(x^3 + x2 + x + 1)$ might lead them to the general formula for the sum of a geometric series. As they work to solve a problem, mathematically proficient students maintain oversight of the process, while attending to the details. They continually evaluate the reasonableness of their intermediate results.

THE MATHEMATICAL PRACTICES FOR AP CALCULUS

The Mathematical Practices for AP Calculus (MPACs) capture important aspects of the work that mathematicians engage in, at the level of competence expected of AP Calculus students. They are drawn from the rich work in the National Council of Teachers of Mathematics (NCTM). Process Standards and the Association of American Colleges and Universities (AAC&U) Quantitative Literacy VALUE Rubric. Embedding these practices in the study of calculus enable students to establish mathematical lines of reasoning and use them to apply mathematical concepts and tools to solve problems. The Mathematical Practices for AP Calculus are not intended to be viewed as discrete items that can be checked off a list; rather, they are highly interrelated tools that should be utilized frequently and in diverse contexts.
(Source: College Board)

MPAC 1 - Reasoning with definitions and theorems.

- Use definitions and theorems to build arguments, to justify conclusions or answers, and to prove results
- Confirm that hypotheses have been satisfied in order to apply the conclusion of a theorem.
- Apply definitions and theorems in the process of solving a problem.
- Develop conjectures based on exploration with technology.
- Produce examples and counterexamples to clarify understanding of definitions, to investigate whether converses of theorems are true or false, or to test conjectures.

MPAC 2 - Connecting concepts.

- Relate the concept of a limit to all aspects of calculus.
- Use the connection between concepts (e.g. rate of change and accumulation) or processes (e.g. differentiation and its inverse process, antidifferentiation) to solve problems.
- Connect concepts to their visual representations with and without technology.
- Identify a common underlying structure in problems involving different contextual situations.

MPAC 3 - Implementing algebraic/computational process.

- Select appropriate mathematical strategies.
- Sequence algebraic/computational procedures logically.
- Complete algebraic/computational processes correctly.
- Apply technology strategically to solve problems.
- Attend to precision graphically, numerically, analytically, and verbally and specify units of measure.
- Connect the results of algebraic/computational process to the question asked.

MPAC 4 - Connecting multiple representations.

- Associate tables, graphs and symbolic representations of functions.
- Develop concepts using graphical, symbolical, or numerical representations with and without technology.

- Identify how mathematical characteristics of functions are related in different representations.
- Extract and interpret mathematical content from any presentation of a function (e.g. utilize information from a table of values.)
- Construct one representational form from another (e.g. a table from a graph, etc.)
- Consider multiple representations of a function to select or construct a useful representation for solving a problem

MPAC 5 - Building notational fluency.
- Know and use a variety of notations
- Connect notation to definitions (e.g. relating the notation for the definite integral to that of the limit of a Riemann sums).
- Connect notation to different representations (graphical, numerical, analytical, and verbal).
- Assign meaning to notation, accurately interpreting the notation in a given problem and across different contexts.

MPAC 6 - Communicating.
- Clearly present methods, reasoning, justifications, and conclusions.
- Use accurate and precise language and notation.
- Explain the meaning of expressions, notation, and results in terms of a context (inc. units).
- Explain the connections among concepts.
- Critically interpret and accurately report information provided by technology.
- Analyze, evaluate, and compare the reasoning of others.

III. Scope & Sequence

Unit 1: VECTORS AND GEOMETRY OF SPACE (10 weeks)

Write the component form of a vector
Perform vector operations and interpret the results geometrically
Write a vector as a linear combination of standard unit vectors
Use vectors to solve problems involving force or velocity
Analyze vectors in space
Use three-dimensional vectors to solve real-life problems
Use properties of the dot product of two vectors
Find the angle between two vectors using the dot product
Find the direction cosines of a vector in space
Find the projection of a vector onto another vector
Find the cross product of two vectors in space
Use the triple scalar product of three vectors in space
Write a set of parametric equations for a line in space
Write a linear equation to represent a plane in space
Sketch the plane given by a linear equation
Find the distance between points, planes and lines in space
Recognize and write equations for cylindrical surfaces, quadratic surfaces, and surfaces of revolution Use cylindrical and spherical coordinate to represent surfaces in space

Unit 2: VECTOR-VALUED FUNCTIONS (8 weeks)

Analyze and sketch a space curve given by a vector-valued function
Extend the concepts of limits and continuity to vector-valued functions
Differentiate and integrate a vector-valued function
Describe the velocity and acceleration associated with a vector-valued function
Use a vector-valued function to analyze projectile motion
Find a unit tangent vector at a point on a space curve
Find the tangential and normal components of acceleration
Find the arc length of a space curve
Use the arc length parameter to describe a plane curve or space curve
Find the curvature of a curve at a point on the curve
Use a vector-valued function to find frictional force

Unit 3: FUNCTIONS OF SEVERAL VARIABLES (6 weeks)

Understand the notation for a function of several variables
Sketch the graph and level curves for a function of two variables
Sketch the level surface for a function of three variables
Use technology to sketch the graph of a function of two variables
Understand the definition of a neighborhood in the plane
Understand the definition of the limit of a function of two variables
Extend the concept of continuity to a function of two or three variables

Find and use a particle derivative of a function of two or three variables
Find the higher-order partial derivatives of a function of two or three variables
Understand the concept of increments and differentials
Extend the concept of differentiability to a function of two variables
Use a differential as an approximation
Use the Chain Rules for functions of several variables
Find partial derivative implicitly
Find and use directional derivatives of a function of two variables
Find the gradient of a function of two variables and use the gradient in applications
Find the directional derivatives and gradients for functions of three variables
Find the equations of tangent planes and normal lines to surfaces
Find the angle of inclination of a plane in space
Compare the gradients of ∇f xy (,) and ∇f xy (,)
Find absolute and relative extrema of a function of two variables
Use the Second Partials Test to find relative extrema of a function of two variables
Solve optimization problems involving functions of several variables
Use the method of least squares
Understand the Method of Lagrange Multipliers
Use Lagrange Multipliers to solve constrained optimization problems

Unit 4: MULTIPLE INTEGRATION (6 weeks)

Evaluate an iterated integral
Use an iterate integral to find the area of a plane region
Use the double integral to represent the volume of solid regions
Use properties of double integrals
Evaluate a double integral as an iterated integral
Write and evaluate double integrals in polar coordinates
Find the mass of planar lamina using a double integral
Find the center of mass of planar lamina using double integrals
Find the moments of inertia using double integrals
Use a double integral to find the area of a surface
Use triple integrals to find the volume of a solid region and in other applications
Write and evaluate a triple integral in cylindrical or spherical coordinates
Understand the concept of a Jacobian and use a Jacobian to change variable in a double integral

Unit 5: VECTOR FIELDS (10 weeks)

Understand the concept of a vector field
Determine whether a vector field is conservative
Find the curl and divergence of a vector field
Understand and use the concept of piecewise smooth curve
Write and evaluate a line integral, a line integral of a vector field, and a line integral in differential form
Understand and use the Fundamental Theorem of Line Integrals

Understand the concepts of independence of path and conservative of energy
Use Green's Theorem to evaluate a line integral
Find a set of parametric equations to represent a surface
Find a normal vector and a tangent plane to a parametric surface
Find the area of a parametric surface
Evaluate a surface integral as a double integral and for a parametric surface
Determine the orientation of a surface
Apply the Divergence Theorem to calculate flux
Understand and use Stoke's Theorem

IV. Unit Descriptions

UNIT 1: VECTORS AND GEOMETRY OF SPACE

Enduring Understanding

Students will understand that...

1. Vectors and coordinate systems for three-dimensional space.
2. Real-Value functions of two variables as a representation of a surface in space
3. Vectors as a way of describing lines and planes in space as well as velocities and accelerations of objects that move through space.
4. Parametric systems are used to represent curves and surfaces.
5. Limits can be used to analyze the behavior of functions of one or more variable.
6. Vectors are used to represent many physical characteristics.
7. An appropriate coordinate system should be selected for each situation.

Essential Questions

1. What is a vector, what are its properties and how does one measure its magnitude and direction?
2. What operations can be performed using vectors?
3. What is a dot product and how can it be used to measure the work done by a force?
4. What is a determinant?
5. How are lines and planes defined in three space?
6. How can you use vectors to determine equations of lines and planes in space?
7. Why are parametric systems often preferable to equation form for representing curves?
8. How are functions in two variables defined and how do they relate to surfaces in 3D-space?
9. How do the rectangular, cylindrical and spherical coordinate systems relate to each other? How would you determine an appropriate coordinate system for a particular situation?
10. How are vectors represented in non-cartesian systems?

Learning Objectives

Students will be able to….

1. Write the component form of a vector, or as a linear combination of standard unit vectors
2. Perform vector operations and interpret the results geometrically
3. Use vectors to solve problems involving force or velocity
4. Analyze vectors in space
5. Use three-dimensional vectors to solve real-life problems
6. Use properties of the dot product of two vectors
7. Find the angle between two vectors using the dot product
8. Find the direction cosines of a vector in space
9. Find the projection of a vector onto another vector
10. Find the cross product of two vectors in space
11. Use the triple scalar product of three vectors in space

12. Write a set of parametric equations for a line in space
13. Write a linear equation to represent a plane in space
14. Sketch the plane given by a linear equation
15. Find the distance between points, planes and lines in space
16. Recognize and write equations for cylindrical surfaces, quadratic surfaces, and surfaces of revolution Use cylindrical and spherical coordinate to represent surfaces in space

Common Core State Standards

- HSN.RN.A: Extend the properties of exponents to rational exponents.
- HSN.RN.B: Use properties of rational and irrational numbers.
- HSN.Q.A: Reason quantitatively and use units to solve problems.
- HSN.VM.A: Represent and model with vector quantities.
- HSN.VM.B: Perform operations on vectors
- HSA.SSE.A: Interpret the structure of expressions
- HSA.SSE.B: Write expressions in equivalent forms to solve problems
- HSA.APR.A: Perform arithmetic operations on polynomials
- HSA.APR.B: Understand the relationship between zeros and factors of polynomials
- HSA.APR.C: Use polynomial identities to solve problems
- HSA.APR.D: Rewrite rational expressions
- HSA.CED.A: Create equations that describe numbers or relationships
- HSA.REI.A: Understand solving equations as a process of reasoning and explain the reasoning.
- HSA.REI.B: Solve equations and inequalities in one variable
- HSA.REI.C: Solve systems of equations
- HSA.REI.D: Represent and solve equations and inequalities graphically
- HSF.IF.A: Understand the concept of a function and use function notation
- HSF.IF.B: Interpret functions that arise in applications in terms of the context
- HSF.IF.C: Analyze functions using different representations
- HSF.BF.A: Build a function that models a relationship between two quantities.
- HSF.BF.B: Build new functions from existing functions
- HSF.LE.A: Construct and compare linear, quadratic and exponential models and solve problems
- HSF.LE.B: Interpret expressions for functions in terms of the situation they model
- HSF.TF.A: Extend the domain of trigonometric functions using the unit circle
- HSF.TF.B: Model periodic phenomena with trigonometric functions
- HSF.TF.C: Prove and apply trigonometric identities
- HSG.SRT.A: Understand similarity in terms of similarity transformations
- HSG.SRT.B: Prove theorems involving similarity
- HSG.SRT.C: Define trigonometric ratios and solve problems involving right triangles
- HSG.SRT.D: Apply trigonometry to general triangles
- HSG.C.A: Understand and apply theorems about circles
- HSG.C.B: Find arc lengths and areas of sectors of circles
- HSG.GPE.B: Use coordinates to prove simple geometric theorems algebraically
- HSG.GMD.B Visualize relationships between two-dimensional and three-dimensional objects
- HSG.MG.A: Apply geometric concepts in modeling situations

Suggested Activities/Modifications

Differentiation strategies may include, but are not limited to, learning centers and cooperative learning activities in either heterogeneous or homogeneous groups, depending on the learning objectives and the number of students that need further support and scaffolding, versus those needing more challenge and enrichment. Modifications may also be made as they relate to the special needs of students in accordance with their Individualized Education Programs (IEPs) or 504 plans, or English Language Learners (ELL). These may include, but are not limited to, extended time, copies of class notes, refocusing strategies, preferred seating, study guides, and/or suggestions from special education or ELL teachers.

New Jersey Common Core State Standards – Technology

- 8.1.12.A – F: All students will use digital tools to access, manage, evaluate, and synthesize information in order to solve problems individually and collaborate and to create and communicate knowledge.

Career Readiness Practices

- CRP1: Act as a responsible and contributing citizen and employee.
- CRP2: Apply appropriate academic and technical skills.
- CRP4: Communicate clearly and effectively and with reason.
- CRP6: Demonstrate creativity and innovation.
- CRP7: Employ valid and reliable research strategies.
- CRP8: Utilize critical thinking to make sense of problems and persevere in solving them.
- CRP9: Model integrity, ethical leadership and effective management.
- CRP10: Plan education and career paths aligned to personal goals.
- CRP11: Use technology to enhance productivity.
- CRP12: Work productively in teams while using cultural global competence.

NJCCCS Standard 9.2 - Career Awareness, Exploration, and Preparation

- 9.2.12.C.1 - Review career goals and determine steps necessary for attainment.
- 9.2.12.C.3 - Identify transferable career skills and design alternate career plans.

NJCCCS Standard 9.3 Career and Technical Education

- 9.3.ST.2 - Use technology to acquire, manipulate, analyze and report data.
- 9.3.ST.5 - Demonstrate an understanding of the breadth of career opportunities and means to those opportunities in each of the Science, Technology, Engineering & Mathematics Career Pathways.
- 9.3.ST-SM.3 Analyze the impact that science and mathematics has on society.
- 9.3.ST-SM.4 Apply critical thinking skills to review information, explain statistical analysis, and to translate, interpret and summarize research and statistical data.

UNIT 2: VECTOR-VALUED FUNCTIONS

Enduring Understanding

Students will understand that...

1. Vector-valued functions as a description of curves and surfaces in space
2. Vector-valued functions as a description of the motion of objects through space.
3. Vector functions are useful in analyzing scalar systems and physical systems.

Essential Question(s)

1. What is a vector function?
2. How do vector functions differ from parametric functions?
3. How do limits of more than one variable differ from limits of one variable?
4. How do limit properties lead to the differences between the differentials or derivatives of functions of one variable and functions of more than one variable?
5. What is a derivative/integral of a vector function?
6. What is a space curve and how do we measure its length and curvature?
7. What are tangent, bi-normal, and normal vectors as well as the normal and osculating planes and how do they relate to a space curve?
8. How can the ideas of a tangent and normal vectors and curvature be used in physics to study the motion of an object, including its velocity and acceleration, along a space curve?
9. What is a parametric surface and how are they defined by parametric vector functions?
10. What physical interpretations can be applied to multivariable derivative?

Learning Objectives

Students will be able to...

1. Analyze and sketch a space curve given by a vector-valued function
2. Extend the concepts of limits and continuity to vector-valued functions
3. Differentiate and integrate a vector-valued function
4. Describe the velocity and acceleration associated with a vector-valued function
5. Use a vector-valued function to analyze projectile motion
6. Find a unit tangent vector at a point on a space curve
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- 9.3.ST-SM.4 Apply critical thinking skills to review information, explain statistical analysis, and to translate, interpret and summarize research and statistical data.

UNIT 3: FUNCTIONS OF SEVERAL VARIABLES

Enduring Understanding

Students will understand that...

1. The concepts of differential calculus as they relate to various real and vector valued functions.
2. Various functions will be explored and understood verbally, numerically, algebraically and visually.
4. Limits can be used to analyze the behavior of functions of one or more variables.
5. There is a relationship between differential quantities and rates of change.
6. Calculus of functions in several variables is the basis of many common analytical techniques that do not involve calculus.
7. There is a relationship between derivatives of functions of one variable to derivatives of functions of multiple variables, but they are not the same thing.

Essential Question(s)

1. What is a contour map and what are level curves and level surfaces?
2. What does it mean for a function in three-space to be continuous?
3. How can you determine if a limit exists in three-space?
4. What is a partial derivative and how is it interpreted?
5. What is a linear or tangent plane approximation of a function at a point and what is it used for?
6. How is the chain rule applied when taking derivatives of functions of two variables?
7. How does one implicitly differentiate a function of three variables?
8. What is a directional derivative?
9. What is a gradient vector and what meaning does it have?
10. How does one calculate the minima and maxima values of a function of two variables?
11. What applications are there for maximizing or minimizing the value of a function?
12. How is integration in multiple dimensions reduced to integration in one dimension?
13. How are bounds interpreted and developed?
14. How is multiple integration used to generalize solutions for previously studied applications?
15. How can varying the order of integration simplify evaluation?

Learning Objectives

Students will be able to:

1. Understand the notation for a function of several variables
2. Sketch the graph and level curves for a function of two variables
3. Sketch the level surface for a function of three variables
4. Use technology to sketch the graph of a function of two variables
5. Understand the definition of a neighborhood in the plane
6. Understand the definition of the limit of a function of two variables
7. Extend the concept of continuity to a function of two or three variables
8. Find and use a particle derivative of a function of two or three variables
9. Find the higher-order partial derivatives of a function of two or three variables
10. Understand the concept of increments and differentials

11. Extend the concept of differentiability to a function of two variables
12. Use a differential as an approximation
13. Use the Chain Rules for functions of several variables
14. Find partial derivative implicitly
15. Find and use directional derivatives of a function of two variables
16. Find the gradient of a function of two variables and use the gradient in applications
17. Find the directional derivatives and gradients for functions of three variables
18. Find the equations of tangent planes and normal lines to surfaces
19. Find the angle of inclination of a plane in space
20. Compare the gradients of ∇ f xy (,) and ∇ f xy (,)
21. Find absolute and relative extrema of a function of two variables
22. Use the Second Partials Test to find relative extrema of a function of two variables
23. Solve optimization problems involving functions of several variables
24. Use the method of least squares
25. Understand the Method of Lagrange Multipliers
26. Use Lagrange Multipliers to solve constrained optimization problems

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- HSN.Q.A: Reason quantitatively and use units to solve problems.
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- HSN.VM.B: Perform operations on vectors
- HSA.SSE.A: Interpret the structure of expressions
- HSA.SSE.B: Write expressions in equivalent forms to solve problems
- HSA.APR.A: Perform arithmetic operations on polynomials
- HSA.APR.B: Understand the relationship between zeros and factors of polynomials
- HSA.APR.C: Use polynomial identities to solve problems
- HSA.APR.D: Rewrite rational expressions
- HSA.CED.A: Create equations that describe numbers or relationships
- HSA.REI.A: Understand solving equations as a process of reasoning and explain the reasoning.
- HSA.REI.B: Solve equations and inequalities in one variable
- HSA.REI.C: Solve systems of equations
- HSA.REI.D: Represent and solve equations and inequalities graphically
- HSF.IF.A: Understand the concept of a function and use function notation
- HSF.IF.B: Interpret functions that arise in applications in terms of the context
- HSF.IF.C: Analyze functions using different representations
- HSF.BF.A: Build a function that models a relationship between two quantities.
- HSF.BF.B: Build new functions from existing functions
- HSF.LE.A: Construct and compare linear, quadratic and exponential models and solve problems
- HSF.LE.B: Interpret expressions for functions in terms of the situation they model
- HSF.TF.A: Extend the domain of trigonometric functions using the unit circle
- HSF.TF.B: Model periodic phenomena with trigonometric functions

- HSF.TF.C: Prove and apply trigonometric identities
- HSG.SRT.A: Understand similarity in terms of similarity transformations
- HSG.SRT.B: Prove theorems involving similarity
- HSG.SRT.C: Define trigonometric ratios and solve problems involving right triangles
- HSG.SRT.D: Apply trigonometry to general triangles
- HSG.C.A: Understand and apply theorems about circles
- HSG.C.B: Find arc lengths and areas of sectors of circles
- HSG.GPE.B: Use coordinates to prove simple geometric theorems algebraically
- HSG.GMD.B Visualize relationships between two-dimensional and three-dimensional objects
- HSG.MG.A: Apply geometric concepts in modeling situations

Suggested Activities/Modifications

Differentiation strategies may include, but are not limited to, learning centers and cooperative learning activities in either heterogeneous or homogeneous groups, depending on the learning objectives and the number of students that need further support and scaffolding, versus those needing more challenge and enrichment. Modifications may also be made as they relate to the special needs of students in accordance with their Individualized Education Programs (IEPs) or 504 plans, or English Language Learners (ELL). These may include, but are not limited to, extended time, copies of class notes, refocusing strategies, preferred seating, study guides, and/or suggestions from special education or ELL teachers.

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Career Readiness Practices

- CRP1: Act as a responsible and contributing citizen and employee.
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- CRP12: Work productively in teams while using cultural global competence.

NJCCCS Standard 9.2 - Career Awareness, Exploration, and Preparation

- 9.2.12.C.1 - Review career goals and determine steps necessary for attainment.
- 9.2.12.C.3 - Identify transferable career skills and design alternate career plans.

NJCCCS Standard 9.3 Career and Technical Education

- 9.3.ST.2 - Use technology to acquire, manipulate, analyze and report data.

- 9.3.ST.5 - Demonstrate an understanding of the breadth of career opportunities and means to those opportunities in each of the Science, Technology, Engineering & Mathematics Career Pathways.
- 9.3.ST-SM.3 Analyze the impact that science and mathematics has on society.
- 9.3.ST-SM.4 Apply critical thinking skills to review information, explain statistical analysis, and to translate, interpret and summarize research and statistical data.

UNIT 4: MULTIPLE INTEGRATION

Enduring Understanding

Students will understand that...

1. Integration is inherently a summing process in one dimension.
2. The idea of the definite integral is extended to double and triple integrals of functions of two and three variables
3. Double and Triple integrals are used to calculate volumes, surface areas, masses, electrical charges and other variable characteristics of a surface.
4. Integrals can be iterated to evaluate systems involving more than one independent variable.
5. Multiple integration is the general tool for problems involving volume and surface areas.

Essential Question(s)

1. What is a double integral and how can it be used to find the volume of a solid?
2. How do you express a double integral as an iterated integral so that we can use standard integration methods to evaluate the expression?
3. How do we integrate a function of a general, non-rectangular, region?
4. How do you use polar coordinates to simplify the integration of solids over circular regions?
5. How can double integrals be used to calculate mass, electrical charge, center of mass, moment of inertia and other physical attributes of a solid?
6. How can double integrals be used to calculate the surface area of a solid?
7. What is a triple integral and how can it be used to evaluate functions of three variables?
8. How can triple integrals be used to calculate various physical attributes of a function of three variables, such as density?
9. How do we evaluate the triple integral of certain solids using cylindrical or spherical coordinates?
10. How can the method of substitution, change of variables, be used to simplify an integral?
11. What is a Jacobian?

Learning Objectives

Students will be able to:

1. Evaluate an iterated integral
2. Use an iterate integral to find the area of a plane region
3. Use the double integral to represent the volume of solid regions
4. Use properties of double integrals
5. Evaluate a double integral as an iterated integral
6. Write and evaluate double integrals in polar coordinates

7. Find the mass of planar lamina using a double integral
8. Find the center of mass of planar lamina using double integrals
9. Find the moments of inertia using double integrals
10. Use a double integral to find the area of a surface
11. Use triple integrals to find the volume of a solid region and in other applications
12. Write and evaluate a triple integral in cylindrical or spherical coordinates
13. Understand the concept of a Jacobian and use a Jacobian to change variable in a double integral

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UNIT 5: VECTOR FIELDS

Enduring Understanding

Students will understand that...

1. Vector fields and spaces are the basis of modern physics, chemistry, engineering and many other topical areas.
2. There is a definite relationship between scalar functions and vector spaces, and this relationship can be used to find solutions to particular problems.
3. There are physical interpretations that can be applied to vector field concepts (such as curl and divergence).

Essential Question(s)

1. What does the sketch of a vector field look like?
2. What is the relation between vector fields and scalar spaces?
3. What is a conservative vector field?
4. How is a piecewise smooth parametrization found, written, evaluated on a line integral, and how does Green's Theorem apply?
5. What is the Fundamental Theorem of Line Integrals?
6. What is a parametric surface, and how do a set of parametric equations represent a surface?
7. How is a normal vector calculated?
8. What is the tangent plane of a surface?
9. What is the area of a surface?
10. How are the Divergence Theorem and Stoke's Theorems applied?

Learning Objectives

Students will be able to:

1. Understand the concept of a vector field
2. Determine whether a vector field is conservative
3. Find the curl and divergence of a vector field
4. Understand and use the concept of piecewise smooth curve
5. Write and evaluate a line integral (of a vector, as well as in differential form)
6. Understand and use the Fundamental Theorem of Line Integrals
7. Understand the concepts of independence of path and conservative of energy
8. Use Green's Theorem to evaluate a line integral
9. Find a set of parametric equations to represent a surface

10. Find a normal vector and a tangent plane to a parametric surface
11. Find the area of a parametric surface
12. Evaluate a surface integral as a double integral and for a parametric surface
13. Determine the orientation of a surface
14. Apply the Divergence Theorem to calculate flux
15. Understand and use Stoke's Theorem

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V. Course Materials (included, but not limited to)

- **Textbook**
  Title: *Multivariable Calculus, 8th Edition*
  Author: Larson, Hostetler, Edwards
  Year: 2006

- **Supplemental Resources**
  The course uses a selection of other multivariable textbooks and online resources, including assessments from various colleges/universities.

- **Technology:**
  TI-Nspire Graphing Calculator
  Google Applications

VI. Assessments (included, but not limited to)

- Do Now Problems
- Quizzes
- Unit Tests
- Classwork
- Homework
- Midterm Exam
- Projects - Research

VII. Cross Curricular Aspects

- Use of Calculator (Sciences)
- Applications of Calculus: (Science, Arts, Business)
- Alternate Assessments (Arts and Humanities)
- Problem-Solution Writing and Discussion: exposition, problem statement, road map, solution, and extension. (Humanities)